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Coordinate Stretching and Interface Location III. A General Relaxation Method with Application to Composite Polytropes

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A general method is developed for the numerical solution of two-point boundary value problems with interfaces. The conventional method of perturbation, discretization and iteration, is rendered uniformly valid throughout the configuration by the application of a generalized Poincaré–Lighthill theory of strained coordinates. Conditions of applicability are discussed. The method is illustrated through application to the problem of a spherical composite polytrope both with and without a density discontinuity. The numerical solutions show good agreement with an analytical solution which has been derived for a particular composite polytrope.

I. INTRODUCTION

Nonlinear two-point boundary value problems (or jury problems [1]) are particularly difficult to solve numerically. To obtain a solution, it is necessary to satisfy the given conditions at each boundary as well as the differential equations throughout the configuration. Thus, information supplied at each boundary must be transferred through the configuration to the other boundary. More difficult problems frequently arise in which there are sudden changes in physical conditions between the boundaries (e.g., shock fronts, or discontinuities in temperature gradient or density). In such cases it is often a satisfactory physical approximation to postulate the existence of interfaces which separate domains within the configuration that are governed by different sets of equations, along with criteria specifying the conditions at those interfaces. In such cases, unknown auxiliary eigenvalues describe the conditions at the interfaces as well as the location of the interfaces within the configuration, and serve to knit the different parts of the configuration together.

Most methods of determining solutions to jury problems make use of trial solutions, with iterative corrections determined by some form of perturbation analysis. However, in most cases, the perturbations are calculated only for the

dependent variables of the system of differential equations, while the independent variable perturbations are ignored. We refer to such formulations as conventional or *C* methods. Thus *C* methods are restrictive, since the trial solution is not permitted to relax through simultaneous changes in all variables.

Generalized approaches to perturbation theory were originated by Lindstedt and Poincaré [2], and extended by Lighthill [3]. In these methods, the independent variables were perturbed (or "stretched") along with the dependent variables. We refer to methods of coordinate stretching as *PL* theory. The *PL* theory has been adapted to jury problems with interfaces by Usher [4], [5 = Paper I] who found that the nonhomogeneous terms of the linearized system of differential equations were dependent on the conditions at the interface. In principle therefore the interface conditions are incorporated into the perturbation scheme by allowing corrections to the independent variable. This is possible because the additional perturbation equation generated by allowing corrections to the independent variable is in general coupled to the constraint at the interface (in addition to the boundary conditions) so that the problem is not underdetermined.

A vastly improved generalized perturbation theory has been formulated by Pritulo [6] and Usher [4], [7 = Paper II], who showed that near-identity transformations in an independent variable can be incorporated in expansions of the dependent variables, with the desirable consequence that the linearized system of perturbation equations is the same as if no coordinate stretching had occurred; i.e., it is identical to the system obtained by the *C* method. The basic concept behind this approach can be traced back to the work of Lord Rayleigh (see e.g. [8]). In the case of ordinary differential equations, it was clear from Paper II that conditions for finite perturbations in the independent variable were evident almost by inspection. Later analysis [9 = Paper III] revealed two cases in which the generalized theory was applicable, and one in which it was not, and the criterion of Wasow [10] was shown to be a special case. In this paper the theory described above will be referred to as the generalized Poincaré–Lighthill or *GPL* theory; it contains the identical features of *PL* theory insofar as the independent variable is perturbed, but differs in an essential way from *PL* theory in the manner in which this perturbation is achieved.

It is frequently the case that the *GPL* formulation is superior to *PL* theory because it is more simple and more convenient to apply, and appears to offer greater insight into the problem at hand [6–9]. It is clear that a solution by the *C* formulation accomplishes the major part of the work, while the complete solution by the *GPL* method requires only a small amount of additional computational or algebraic labor; thus the *GPL* method first accomplishes what the *C* method does, but allows for the option of an independent variable perturbation which can then be used if necessary or desirable. As expected, both the *GPL* and the *PL* methods reduce to the *C* method when the independent variable is not perturbed.

In this paper we formulate a relaxation method for the numerical solution of a general jury problem with interfaces by the use of the *GPL* perturbation theory, and apply it to the problem of composite polytropes. The *GPL* theory is developed in Part II, and necessary conditions for applicability are examined in Part III. In Part IV we adapt the method to the problem of a spherical composite polytrope both with and without a density discontinuity at the interface. The numerical solutions are checked by comparing them with the analytic solution given in Appendix A. Part V is a presentation and discussion of the results.

II. A Generalized Relaxation Method

A. *Introductory Remarks*

On perturbing the non-linear differential system

$$dv/du = f(v, u), \tag{1}$$

by means of the expansions

$$v(u) = v_0(u_0) + \delta\tilde{v}(u_0) + \delta u(u_0) f_0(v_0 , u_0), \tag{2}$$

$$u = u_0 + \delta u(u_0), \tag{3}$$

and neglecting terms of second and higher order, it has been shown in Paper II that the linearized first order system becomes

$$(dv_0/du_0) + (d\delta\tilde{v}/du_0) = f_0 + \delta\tilde{v}^j(\partial f/\partial v^j)_0 , \tag{4}$$

where the nonhomogeneous error term can be written

$$E_0 = f_0 - (dv_0/du_0). \tag{5}$$

Here zero subscripts refer to trial values and repeated superscripts denote summation. Equation (4) is the basis of the *GPL* method, and is not now explicitly dependent on the coordinate perturbation of the independent variable as is Eq. (6) of Paper I. Equation (4) is in fact identical in form to the perturbation equations derived by the *C* method, as is clear on setting $\delta u \equiv 0$ in Eqs. (2) and (3). This equivalence is expressed by Eq. (18) of Paper III.

A corresponding relationship between the first order boundary conditions in the *C* and *GPL* methods can also be achieved [7, 9] if we require that f_0 be everywhere finite in the normalized range of the independent variable

$$0 \leqslant u \leqslant 1. \tag{6}$$

Let the jury problem have known boundary values

$$v_0^j(0) = v^j(0); \ (j = 1, 2,..., m), \tag{7}$$

$$v_0^j(1) = v^j(1); \ (j = m + 1, m + 2,..., M). \tag{8}$$

Eq. (6) requires that

$$\delta u(0) = \delta u(1) = 0, \tag{9}$$

and therefore it follows from Eqs. (2), (3), (7), (8) and (9) that

$$\delta \tilde{v}^j(0) = 0; \ (j = 1, 2,..., m), \tag{10}$$

$$\delta \tilde{v}^j(1) = 0; \ (j = m + 1, m + 2,..., M). \tag{11}$$

Equations (10) and (11) are the *GPL* method boundary conditions which are clearly equivalent to the *C* method conditions for which $\delta u \equiv 0$. Equations (4), (10) and (11) comprise the system to be solved in the first step according to the *GPL* method, and are formally equivalent to solving the problem by the *C* method. The second step of the *GPL* method incorporates the information given at the interface, as described in Sections II C, D below.

B. *Discretization*

We can compute the solution to Eqs. (4), (10) and (11) by replacing Eq. (4) by a system of linear algebraic equations which can be solved for example by triangular decomposition with pivotal interchanges [11]. Following Fox [12], equations of the form (1) can be written

$$(v_{i+1} - v_i)/(u_{i+1} - u_i) = \tfrac{1}{2}(f_{i+1} + f_i), \tag{12}$$

which is accurate to terms of order $\frac{1}{12}(u_{i+1} - u_i)^2 \, d^3v(v)/du^3$ where $u_i < v < u_{i+1}$. Thus Eq. (4) becomes

$$\begin{aligned}(\delta \tilde{v}_{i+1}^j - \delta \tilde{v}_i^j)/(u_{0,i+1} - u_{0,i}) = {} & \tfrac{1}{2}[\delta \tilde{v}_{i+1}^k (\partial f_{i+1}^j/\partial v_{i+1}^k)_0 + \delta \tilde{v}_i^k (\partial f_i^j/\partial v_i^k)_0] \\ & + \tfrac{1}{2}(f_{i+1}^j + f_i^j)_0 - (v_{0,i+1}^j - v_{0,i}^j)/(u_{0,i+1} - u_{0,i}),\end{aligned} \tag{13}$$

where $j = 1, 2,..., M$, and subscripts $i = 1, 2,..., N$ refer to the interstitial points of the configuration. The last two terms of Eq. (13) give the error term of Eq. (5). Equation (13) can be obtained either (i) by perturbing Eq. (1) to give Eq. (4) and then forming the difference equations, or (ii), by forming difference equations (12) directly from Eq. (1) and then perturbing them. Moreover it can be shown that the same difference Eq. (13) results from either the *GPL* method or the *C* method. Of

course in the C method the variable u remains unchanged from iteration to iteration (i.e., the o subscript on u is redundant). Therefore it is only with the incorporation of the interface conditions that differences in the C and GPL methods appear. (Section IIC below)

At the two boundaries it follows from the given conditions (10) and (11) that

$$\delta\tilde{v}_1^{\,j} = 0, \qquad (j = 1, 2,..., m), \tag{14}$$

$$\delta\tilde{v}_{N+1}^{j} = 0, \qquad (j = m + 1, m + 2,..., M). \tag{15}$$

Equations (13)–(15) are $MN + M$ linear equations for the solution of the $M(N + 1)$ unknowns $\delta\tilde{v}_i^{\,j}$, $(j = 1, 2,..., M; i = 1, 2,..., N + 1)$. The development up to this point concludes the formulation of the first step of the GPL method.

C. *Interface Conditions*

At each interface a total of $M + 2$ conditions are available which relate the variables at one side of the interface to the other. Of these conditions, M are needed for the dependent variables, and one for the independent variable. The remaining condition specifies where the transition from one domain of the configuration to the other is to occur; i.e., it is a constraint which governs the location of the interface.

All $M + 2$ conditions can be taken into account in the GPL method. At any interface we have the conditions on continuity

$$\alpha_I^{\,j}(v, u) = \alpha_{I+1}^{j}(v, u), \qquad (j = 1, 2,..., M + 1), \tag{16}$$

and the constraint

$$\sigma_I(v, u) = \sigma_{I+1}(v, u), \tag{17}$$

where the $i = I$ and $i = I + 1$ subscripts denote the side of the interface facing toward the $u = 0$ and $u = 1$ boundaries respectively. For any component of $F(v, u) = \{\alpha^1, \alpha^2,..., \alpha^{M+1}, \sigma\}$ we can write

$$(dF^j/du)_0 = (\partial F^j/\partial u)_0 + f_0^{\,k}(\partial F^j/\partial v^k)_0\,,$$

which from Eq. (4) is true to terms of first order. On applying the perturbation Eqs. (2) and (3) to (16) and (17), we obtain linearized algebraic equations for the first order corrections

$$\delta\tilde{v}_I^{\,k}(\partial\alpha_I^{\,j}/\partial v^k)_0 - \delta\tilde{v}_{I+1}^{k}(\partial\alpha_{I+1}^{j}/\partial v^k)_0 + \delta u_I(d\alpha_I^{\,j}/du)_0 - \delta u_{I+1}(d\alpha_{I+1}^{j}/du)_0$$
$$= \Delta\alpha_0^{\,j}, \qquad (j = 1, 2,..., M + 1) \tag{18}$$

and

$$\delta\tilde{v}_I^{\,k}(\partial\sigma_I/\partial v^k)_0 - \delta\tilde{v}_{I+1}^{k}(\partial\sigma_{I+1}/\partial v^k)_0 + \delta u_I(d\sigma_I/du)_0 - \delta u_{I+1}(d\sigma_{I+1}/du)_0 = \Delta\sigma_0\,, \tag{19}$$

where

$$\Delta\alpha_0^{\,j} = \alpha_{0,I+1}^{j} - \alpha_{0,I}^{j}\,, \tag{20}$$

and

$$\Delta\sigma_0 = \sigma_{0,I+1} - \sigma_{0,I}\,. \tag{21}$$

D. *A Common Case*

Equations (18) and (19) are the necessary $M + 2$ linearized conditions at the interface. They can be simplified for the common case for which the independent variable u is governed by a condition of continuity (for example, in a geometric variable). Then one of the components of Eq. (16)(say the $M + 1$ component) is simply

$$u_I = u_{I+1}\,. \tag{22}$$

The corresponding component of equation (18) is $\delta u_I - \delta u_{I+1} = u_{0,I+1} - u_{0,I}$ which is zero when u is continuous across the interface, so that

$$\delta u_I = \delta u_{I+1}\,. \tag{23}$$

With the help of Eq. (23), it follows that Eqs. (18) and (19) are

$$\delta\tilde{v}_I^{\,k}(\partial\alpha_I^{\,j}/\partial v^k)_0 - \delta\tilde{v}_{I+1}^{k}(\partial\alpha_{I+1}^{j}/\partial v^k)_0 = \Delta\alpha_0^{\,j} + \delta u_I(d\Delta\alpha^j/du)_0\,, \tag{24}$$

for ($j = 1, 2,..., M$), and

$$\delta\tilde{v}_I^{\,k}(\partial\sigma_I/\partial v^k)_0 - \delta\tilde{v}_{I+1}^{k}(\partial\sigma_{I+1}/\partial v^k)_0 = \Delta\sigma_0 + \delta u_1(d\Delta\sigma/du)_0\,, \tag{25}$$

where in conformity with Eqs. (20) and (21)

$$d\Delta\alpha^j/du = (d\alpha_{I+1}^{j}/du) - (d\alpha_I^{\,j}/du), \tag{26}$$

and

$$d\Delta\sigma/du = (d\sigma_{I+1}/du) - (d\sigma_I/du). \tag{27}$$

Equation (25) gives information on the correction to the interface location, and in fact can be solved for δu_I ; we have

$$\delta u_I = [\delta\tilde{v}_I^{\,k}(\partial\sigma_I/\partial v^k)_0 - \delta\tilde{v}_{I+1}^{k}(\partial\sigma_{I+1}/\partial v^k)_0 - \Delta\sigma_0]/(d\Delta\sigma/du)_0\,, \tag{28}$$

which when eliminated from Eq. (24) gives

$$\delta\tilde{v}_I^k \left(\frac{\partial\alpha_I^j}{\partial v^k} - \frac{\partial\sigma_I}{\partial v^k}\frac{d\Delta\alpha^j}{d\Delta\sigma}\right)_0 - \delta\tilde{v}_{I+1}^k \left(\frac{\partial\alpha_{I+1}^j}{\partial v^k} - \frac{\partial\sigma_{I+1}}{\partial v^k}\frac{d\Delta\alpha^j}{d\Delta\sigma}\right)_0$$
$$= \Delta\alpha_0{}^j - \Delta\sigma_0(d\Delta\alpha_0{}^j/d\Delta\sigma_0), \qquad (j = 1, 2,..., M); \tag{29}$$

here we have let

$$\frac{d\Delta\alpha_0{}^j}{du_0} \Big/ \frac{d\Delta\sigma_0}{du_0} = \frac{d\Delta\alpha_0{}^j}{d\Delta\sigma_0}. \tag{30}$$

Equation (29) comprises M equations which connect the part of the configuration facing one side of an interface to the part facing the other side. When combined with the M Eqs. (13) at each of the remaining interstitial points and the M boundary conditions (14) and (15), it constitutes a system of $M(N + 1)$ linear algebraic equations for the determination of $\delta\tilde{v}_i{}^j (j = 1, 2,..., M; i = 1, 2,..., I; I + 1,..., N + 1)$. All explicit coordinate perturbations in u have now been removed and incorporated implicitly in $\delta\tilde{v}_i{}^j$. The first stage of the *GPL* method is thus formally equivalent to the *C* method. The overall procedure and the manner of coordinate stretching is discussed in Section IIF below.

E. *A Special Case*

A special case arises when the σ condition (17) is a function of u only, for then Eq. (17) gives the numerical value of u_I straightaway, and the location of the interface is known. Then only the α conditions (16) need to be used, and there is no need for the second step of the *GPL* method. Thus the *GPL* method reduces to the *C* method in this case.

F. *Method*

A trial solution $v_{0,i}^j(u_0)$ is tested for the occurrence of interfaces according to the σ condition (17), and the Ith interstitial point is then located. In the *GPL* method this point is identified with the interface during all subsequent iterations. (In the *C* method this is generally not possible without the introduction of special tricks). Having solved for all the $\delta\tilde{v}_i{}^j$ including $\delta\tilde{v}_I{}^j$ and $\delta\tilde{v}_{I+1}^j$, $(j = 1, 2,..., M)$, we can find δu_I and δu_{I+1} from Eqs. (23) and (28). This gives the amount by which the independent variable must be stretched at the interstitial point $i = I$, i.e., it gives the corrected location of the interface $u_I = u_{0,I} + \delta u_I$ by Eq. (3). Clearly, all other interstitial points must also be relocated. This can be simply accomplished by prorating the amount of stretching throughout the configuration. For example, when the σ condition (17) indicates the existence of a single interface, we can let

$$\delta u_i = (\delta u_I/u_{0,I})\, u_{0,i}, \qquad (1 \leqslant i \leqslant I), \tag{31}$$

and

$$\delta u_i = \{\delta u_{I+1}/(u_{0,I+1} - 1)\}(u_{0,i} - 1), \qquad (I + 1 \leqslant i \leqslant N + 1) \tag{32}$$

in which by Eqs. (22) and (23) we have of course $\delta u_I = \delta u_{I+1}$ and $u_{0,I} = u_{0,I+1}$. As described in Paper I, this is tantamount to assuming a Hooke's Law of stretching for the independent variable. The assumption of a linear stretching law is quite arbitrary, and other laws can be assumed as long as they conform to the boundary and interface conditions (9) and (23), and condtion (34) below; however the linear form of Eqs. (31) and (32) is consistent with the philosophy of complete linearization in all other aspects of the problem.

All values of $\delta\tilde{v}_i^j$, $\delta u_i (j = 1, 2,..., M; i = 1, 2,..., N + 1)$ are now known and Eqs. (2) and (3) give the corrected solution $v_i^j(u_i)$. It is important to note that while the i-values refer formally to the same interstitial points, these are in general no longer in the same places, nor are the intervals $(u_i - u_{i+1})$ equal. The v_i^j solution now becomes the new trial solution $v_{0,i}^j$ and the process is repeated. If convergence occurs, the final iteration gives the interface eigenvalues u_I and v_I^j in addition to the boundary eigenvalues.

III. Conditions of Applicability

A. *Regions of Uniformity*

Consider problems for which one or more of the v_0^j components are discontinuous at a point $u_{0,I}$. The C method breaks down in the vicinity of $u_{0,I}$, because the corresponding f_0^j components will generally be infinite or discontinuous at that point, and the Taylor series expansions are no longer valid. However the C method can be rendered uniformly valid at all points if we regard the domain $0 \leqslant u_0 \leqslant 1$ as broken up into two parts $0 \leqslant u_0 \leqslant u_{0,I}$ and $u_{0,I+1} \leqslant u_0 \leqslant 1$ for an interface at $u_{0,I} = u_{0,I+1}$. In each domain f_0 is now well-behaved by assumption, and the separate domains are linked at the interface by the α and σ conditions (16) and (17).

For present purposes we regard interfaces as vanishingly small regions over which one or more of v^j of f^j are discontinuous. In general we define a domain of interest to be that region between interfaces (or from a boundary to an interface) over which all v^j and f^j are zero or finite and continuous. For brevity we refer to the domain of interest as D. Thus the C method requires that

$$f_0 \text{ must be continuous and finite or zero in } D. \tag{33}$$

B. *Necessary conditions for the general method*

Condition (33) is also necessary for the applicability of the *GPL* method, because of the formal equivalence between the differential equations and boundary conditions for δv and $\delta\tilde{v}$. However additional conditions are readily deduced from Eqs. (2) and (3). We require that both $\delta u(u_0)$ and $\delta u(u_0) f_0(v_0, u_0)$ be continuous and finite or zero in D in order that the new trial solution $v(u)$ shall be well-behaved

after applying the second step of the *GPL* method. Since condition (33) already specifies conditions on f_*, we have in addition that

$\delta u(u_0)$ must be continuous and finite or zero in D. (34)

By Section IID, the functional dependence of δu on u_0 is arbitrary, but it must be chosen to satisfy condition (34).

C. *Discussion*

The coordinate perturbation δu can be used for a variety of purposes, and we note briefly a difference between the numerical emphasis of this paper and the emphasis of Paper III on asymptotic developments. In Paper III the basic motivation for the derivation of conditions of applicability is still condition (34), but the purpose of the δu function is to avoid increases in the order of the singularities as the perturbation approximation is carried to higher orders, thereby rendering the first order series uniformly valid in D. Thus the choice for δu depends on v_0, δv and f_0; its functional dependence is the prime consideration, while the constant that appears in the expression for δu is determined by the boundary conditions.

In the present case, δu must also be introduced to render the method (of perturbation, discretization and iteration) uniformly valid in the vicinity of an interface. This is necessary because an interface defines a change in the governing equations, a discontinuity in the dependent variables, or even in extreme cases a delta function in their derivatives. These conditions can be more satisfactorily dealt with when an interstitial point occurs at the trial interface during all iterations, which in general is not the case in the *C* method. The use of δu is especially important to avoid a difference approximation across a delta function in a derivative. An added benefit is the extra flexibility that allows the σ condition to be incorporated into the difference equations.

We note that considerations on stability and convergence of the iterative scheme built upon Eqs. (13)–(15), are the same as in the *C* method, in view of the equivalences discussed in Sections IIA and B. An analysis of the affect of incorporating the α and σ conditions (16) and (17), on stability and convergence is beyond the scope of this paper.

As a first test of the *GPL* method, we apply it in the next section to a problem in which one or two of the v^j components are discontinuous at the same interface.

IV. Composite Polytropes

A. *Boundary and Interface Conditions*

The existence theorem for composite, spherical polytropes as formulated by Chandrasekhar [13] states that an equilibrium configuration of prescribed mass m_*

and radius R can be constructed which consists of a core of polytropic index n_C surrounded by an envelope of index n_E, such that the core extends to a prescribed fraction q of the radius. If we choose the radius r to be the independent variable then the σ condition (17) becomes simply $r = qR$ which is one of the special cases discussed in Section IIE. Thus the problem as formulated by Chandrasekhar will not appropriately illustrate the *GPL* method if r is taken as the independent variable; however, we can either choose the mass fraction m as the independent variable, or we can keep r as the independent variable but replace the σ condition of Chandrasekhar by one requiring that the core mass extend to a prescribed fraction ϕ of the total mass. For reasons of convenience we retain r as the independent variable and let the σ condition (17) be

$$m_I = \phi m_* . \tag{35}$$

The α conditions (16) are that the radius r, mass fraction m and pressure p be continuous across the interface, and for the sake of generality we consider configurations for which the density ρ is discontinuous. Thus we let

$$r_{C,I} = r_{E,I}\,; \quad m_{C,I} = m_{E,I}\,; \quad p_{C,I} = p_{E,I}\,; \quad \rho_{C,I} = \psi\rho_{E,I}\,. \tag{36}$$

Conditions (35) and (36) are the $M + 2$ conditions at the interface required for a system of differential equations of order $M = 3$. In addition $M = 3$ boundary conditions must be assigned as the end points $r = 0, R$. These are

$$r = 0; \qquad m = 0; \tag{37}$$

and

$$r = R; \qquad m = m_*\,, \qquad \rho = 0. \tag{38}$$

B. *Equations and transformations*

The system of equations of order $M = 3$ governing the structure of the polytrope in a domain of interest D is

$$dp/dr = -Gm\rho/r^2, \tag{39}$$

$$dm/dr = 4\pi r^2\rho, \tag{40}$$

$$dp/d\rho = \left(1 + \frac{1}{n}\right) p/\rho. \tag{41}$$

Eq. (41) can be integrated to give

$$p = K\rho^{1+1/n}, \tag{42}$$

where

$$K = \textit{constant in } D. \tag{43}$$

The determination of K in each domain D is part of the assigned problem; K_C and K_E are eigenvalues of the problem which in general are not equal when there are discontinuities in n and ρ.

We can use Eq. (42) to eliminate p from Eq. (39), and Eq. (43) can replace Eq. (41). We could also rewrite Eq. (43) as

$$dK/dr = \delta(r - r_I), \qquad (0 \leqslant r \leqslant R) \tag{44}$$

where the δ-function is the derivative of the step function in K. In view of the discussion in section IIIA, we simplify the notation by writing

$$dK/dr = 0, \tag{45}$$

where it is understood that Eq. (45) is valid only in one or other domain $0 \leqslant r < r_I$ or $r_I < r \leqslant R$. Eqs. (39), (40) and (45) comprise the system to be solved. It should be noted that even when ρ is discontinuous, its gradient is not a delta function at the interface; it follows from Eqs. (39) and (40) that $d\rho/dr = -Gm\rho^2/(1 + (1/n))\, pr^2$ which depends on the continuous functions m, p and r, but in general $\rho^2/(1 + (1/n))$ is discontinuous at the interface when ρ and n are discontinuous.

We let

$$r = Rx,\ m = m_* y,\ \rho = (m_*/4\pi R^3)\, z^n, \tag{46}$$

and Eqs. (39), (40) and (45) become in D

$$dy/dx = x^2 z^n, \tag{47}$$

$$dz/dx = -ey/x^2, \tag{48}$$

$$de/dx = 0, \tag{49}$$

where the transformed eigenvalue is

$$e = (4\pi)^{1/n}\, Gm_*^{1-1/n} R^{3/n-1}/K(1 + n). \tag{50}$$

Conditions (35)–(38) becomes with the help of Eqs. (42), (46) and (50)

$$y_I = \phi \tag{51}$$

$$x_{C,I} = x_{E,I} \tag{52}$$

$$y_{C,I} = y_{E,I} \tag{53}$$

$$z_{C,I}^{n_C} = \psi Z_{E,I}^{n_E} \tag{54}$$

$$z_{C,I}/e_C(1 + n_C) = z_{E,I}/e_E(1 + n_E)\, \psi \tag{55}$$

and

$$x = 0; \qquad y = 0, \tag{56}$$

$$x = 1; \qquad y = 1, z = 0, \tag{57}$$

where Eq. (55) is obtained with the help of Eq. (54). Eqs. (47)–(49) and (51)–(57) are the transformed problem.

C. *Applicability Conditions and Numerical Accuracy*

As $x \to 0$, z is finite and $y \propto x^3$, and hence by Eqs. (47)–(49), condition (33) is satisfied throughout $0 \leqslant x \leqslant x_I$. At $x = 1$, $z \to 0$, and $y \to 1$ and condition (33) is again satisfied throughout $x_{I+1} \leqslant x \leqslant 1$. We shall let

$$\delta x_i = (\delta x_I / x_{0,I})\, x_{0,i}\,, \quad (0 \leqslant x_{0,i} \leqslant x_{0,I}), \tag{58}$$

and

$$\delta x_i = \{\delta x_{I+1}/(x_{0,I+1} - 1)\}(x_{0,i} - 1), \quad (x_{0,I+1} \leqslant x_{0,i} \leqslant 1), \tag{59}$$

by analogy with Eqs. (31) and (32), so condition (34) is satisfied. Thus the *GPL* method is necessary (cf. the discussion of section IVA) and it is also applicable for the particular choice of variables made in sections IVA and B.

Also with this particular choice of variables, the functions d^3v/du^3 will not in general all be finite or zero at the boundaries, but since they are calculated only near the interstitial points, the approximation (12) will be valid up to a certain degree of accuracy. The numerical accuracy can be gauged for the illustrative example of this paper since an exact solution is available (Appendix A). Greater numerical accuracy can be obtained by a more complicated choice of variables $v^j(u)$ such that their third derivatives are always finite or zero at the boundaries, or by a more accurate difference approximation. However the above approach is adequate for present purposes, since the present goal is to develop and test the *GPL* method with the minimum of extraneous complexity.

V. Results and Discussion

A. *The GPL Method*

We illustrate the *GPL* method for a number of composite polytropes of center and surface indices 5 and 1, and we select the results for two models as being fairly representative. The first model (model A) has a continuous density distribution ($\psi = 1$ in equations (54) and (55)) while the second model (model B) has about a 26% discontinuity in density. Both models are discontinuous in e. The parameters $\{n_C\,, n_E\,, \gamma, \psi, e_E\,, e_C\,, \phi, k, x_I$ and $z_C(x = 0)\}$ for the exact solutions from

Appendix A are; model A {5, 1, 2.77805, 1, 2.18928, 0.65484, 0.89749, 2.07353, 0.64613, 3.03360} and model B {5, 1, 3.22288, 1.26586, 1.42651, 0.60073, 0.90500,

basis for producing initial trial solutions in the following manner; for each trial solution all the eigenvalues $z_C(x = 0)$, e_E, e_C, x_I and the values of y_i and z_i at the $N + 1$ interstitial points x_i were chosen to be in error by an amount ranging from about 1 % to 30 % (with due regard for the known boundary conditions). A computer program accomplished the iterations for improved solutions (sections IIB and F). The iterations were terminated upon satisfying a convergence criterion that the absolute magnitude of the largest change after any iteration in all variables be less than 10^{-4}. The program was run with values of $N = 20$, 40 and 60.

The demonstration of the *GPL* method is given in Tables I, II and III. For model A, Table I gives the number of iterations required for convergence for 1 %,

TABLE I

The Number of Iterations to Convergence for Model $A(\psi = 1)$.

	Trial solution in error by about			
N	1 %	3 %	10 %	30 %
20	3	4	5	9
40	3	3	5	9
60	3	3	5	9

TABLE II

Differences (in the Sense, Exact-iterated) for the Eigenvalues x_I, e_E and e_C and Maximum Differences in y and z for Model A ($\psi = I$). Iteration 0 Refers to the Trial Solution. Decreases of 0.1 in Δe_E for Iterations 1–6 are Due to Scaling.

Iteration	Δx_I	Δe_E	Δe_C	$\Delta y_{\max}$	$\Delta z_{\max}$
0	0.1461	0.6493	0.2048	−0.1532	0.9336
1	0.1118	0.5493	0.1516	−0.1114	0.7013
2	0.0811	0.4493	0.1033	−0.0767	0.4758
3	0.0542	0.3493	0.0633	−0.0519	0.2687
4	0.0318	0.2493	0.0355	−0.0361	0.0925
5	0.0152	0.1493	0.0199	−0.0251	−0.0367
6	0.0038	0.0493	0.0076	−0.0112	−0.0449
7	0.0002	0.0013	0.0004	−0.0008	−0.0077
8, 9	0.0003	0.0036	0.0008	+0.0016	−0.0049

3%, 10% and 30% trial solutions with $N = 20$, 40 and 60. As might be expected the required number of iterations increases with increasing error in the trial solution and in one case decreases with increasing number of points.

For model A (no density discontinuity, $\psi = 1$), the improvement in the solution after each iteration is shown in Table II in the case of the 30%, $N = 60$ trial solution of Table I. The quantities tabulated are the absolute errors in the eigenvalues x_I, e_E and e_C, along with the maximum errors in y and z. Because of the nature of the boundary values, $\Delta y_{\max}$ usually occurred near $x = 0.5$ (because y is constrained at both boundaries) while $\Delta z_{\max}$ occurred near $x = 0$ (because z is constrained only at $x = 1$). The eigenvalues e_E and $z_C(x = 0)$ are the slowest to converge, since (i) $z_C(x = 0)$ is not constrained by the boundary values, and (ii) Δz_C and Δe_E are directly related through equations (A10) and (A18) for given k and ψ, and for $x_I \approx 0.5$.

TABLE III

As in Table II, but for Model $B(\psi = 1.26586)$.

Iteration	Δx_I	Δe_E	Δe_C	$\Delta y_{\max}$	$\Delta z_{\max}$
0	0.1616	0.1265	0.1007	−0.1994	1.0876
1	0.1340	0.1214	0.0527	−0.1618	0.8847
2	0.1061	0.1115	0.0147	−0.1274	0.7785
3	0.0767	0.0782	−0.0007	−0.0932	0.6121
4	0.0377	−0.0088	−0.0235	−0.0725	0.2952
5	0.0050	−0.0450	−0.0343	−0.0117	0.0852
6	−0.0004	−0.0015	−0.0008	+0.0019	−0.0142
7, 8	−0.0003	−0.0015	−0.0005	+0.0017	−0.0139

For model B (density discontinuity about 26%) the improvement in the solution after each iteration is shown in Table III for a 30% trial solution with $N = 60$. The characteristics of the errors listed in Table III are similar to those of Table II discussed above.

B. *The C Method*

The C method is clearly inapplicable to this composite polytrope problem even in the density continuity case ($\psi = 1$) because of the existence of a delta function in the derivative of the variable e. Consequently special computational devices [12] (e.g. interpolation or the insertion of pseudo-interfaces) must be used. We found that when successful, these devices introduced significantly greater complexity into the computational program; in addition we found convergence and final accuracy to be sensitive to the details and nature of the device.

A pure application of the C method was also attempted for model A in which no attempt was made to locate the interface between interstitial points. Rather, the σ condition (35) was used after every iteration to find the interval in which the transition from core to envelope takes place, and the difference equations were applied across that interval as if the interface occurred at the nearest interstitial point. For $N = 60$, a trial solution in error by about 1 % appeared to converge nicely, but unfortunately the error in $z(x = 0)$ in the converged solution was 17 %, with commensurately large errors in the other eigenvalues. A comparison between the C and GPL methods for problems with step functions (rather than δ functions) in one or more derivatives, will be the subject of a later investigation.

APPENDIX A: An Exact Composite Polytrope

It can be shown by differentiation and substitution that

$$y_C = (3/e_C)^{5/4}\,(k^5x^3/3(1 + k^4x^2)^{3/2}), \tag{A1}$$

$$z_C = (3/e_C)^{1/4}\,(k/(1 + k^4x^2)^{1/2}), \tag{A2}$$

for $0 \leqslant x \leqslant x_I$, and

$$y_E = e_E^{-1/2}\sin[e_E^{1/2}(1 - x)] + x\cos[e_E^{1/2}(1 - x)], \tag{A3}$$

$$z_E = e_E^{1/2}x^{-1}\sin[e_E^{1/2}(1 - x)], \tag{A4}$$

for $x_I \leqslant x \leqslant 1$, is an exact solution to Eqs. (47)–(49), (56) and (57) in the case $(n_C, n_E) = (5, 1)$ where k is an integration constant. The interface conditions (51) and (53)–(55) are

$$\phi = y_{E,I} \tag{A5}$$

$$y_{C,I} = y_{E,I} \tag{A6}$$

$$z_{C,I}^5 = \psi z_{E,I} \tag{A7}$$

$$z_{C,I}e_C^{-1} = \frac{3}{\psi}\,z_{E,I}e_E^{-1} \tag{A8}$$

and in addition at the interface x_I, Eq. (52) is

$$x_{C,I} = x_{E,I} = x_I, \qquad (0 < x_I < 1). \tag{A9}$$

When Eq. (A9) holds, then the four conditions (A5)–(A8) enable the four constants k, x_I, e_C and e_E to be determined when ϕ and ψ are specified, although in general,

transcendental equations must be solved. Alternatively we may regard ϕ as unknown, and one convenient combination of k, x_I, e_C, e_E and ψ as given. If we let

$$\gamma = k^2 x_I > 0, \tag{A10}$$

then exact expressions for k, x_I, e_C, e_E and ϕ can be found in terms of γ and ψ. For convenience we let

$$\Gamma = \frac{3\gamma}{1+\gamma^2} > 0, \tag{A11}$$

which has a maximum value of $\frac{3}{2}$ at $\gamma = 1$.

When $x = x_I$, equations (A5)–(A8) can be rewritten with the help of equations (A1)–(A4). Then, on dividing equations (A7) and (A8) we obtain

$$\psi x_I e_E^{1/2} = \Gamma. \tag{A12}$$

Since ψ, x_I and Γ are all > 0 by Eqs. (A7), (A9) and (A11), we have $e_E^{1/2} > 0$. On dividing Eqs. (A6) and (A8) we obtain

$$[(3-\psi)\gamma^2 - \psi]\sin[e_E^{1/2}(1-x_I)] = 3\gamma\cos[e_E^{1/2}(1-x_I)], \tag{A13}$$

which using trigonometric identities and collecting terms gives an expression for $\tan e_E^{1/2}$ from which

$$e_E^{1/2} = \tan^{-1}\left\{\frac{(\psi-(3-\psi)\gamma^2)\sin(\Gamma/\psi) - 3\gamma\cos(\Gamma/\psi)}{(\psi-(3-\psi)\gamma^2)\cos(\Gamma/\psi) + 3\gamma\sin(\Gamma/\psi)}\right\} > 0. \tag{A14}$$

On squaring Eq. (A13) and changing $\cos^2$ to $1 - \sin^2$, it follows that

$$\sin[e_E^{1/2}(1-x_I)] = 3\gamma/\{[(3-\psi)^2\gamma^2 - \psi]^2 + 9\gamma^2\}^{1/2} > 0 \tag{A15}$$

and thus $\cos[e_E^{1/2}(1-x_I)]$ can be found from Eq. (A13). The inequality in Eq. (A15) follows from the inequalities above and the occurrence of the radical in the expression (A16) for ϕ which is positive. The function (A15) has a maximum of $3/(1+3\psi-\psi^2)$ at $\gamma = [\psi/(3-\psi)]^{1/2}$ and varies as $3/(3-\psi)\gamma$ as $\gamma \to \infty$. Thus for example when $1 \leqslant \psi \leqslant 2$, we have $0 \leqslant \sin[e_E^{1/2}(1-x_I)] \leqslant 1$ for all $\gamma > 0$.

Eqs. (A3), (A5) and (A12) give

$$\phi = 9\gamma^3/\psi e_E^{1/2}(1+\gamma^2)\{[(3-\psi)^2\gamma^2 - \psi)]^2 + 9\gamma^2\}^{1/2} > 0. \tag{A16}$$

From Eqs. (A8), (A12) and (A15) we find

$$e_C = e_E^{1/5}\psi^{2/5}\{[(3-\psi)^2\gamma^2 - \psi]^2 + 9\gamma^2\}^{2/5}/3(1+\gamma^2)^{4/5} > 0. \tag{A17}$$

Thus Eqs. (A10), (A12), (A14), (A16) and (A17) give k, x_I, e_E, ϕ, and e_C parametrically in terms of γ and ψ from which all other quantities can be found. In particular

$$z_C(x = 0) = e_E^{1/5}\psi^{2/5}(1 + \gamma^2)^{7/10}/\{(3 - \psi)^2\gamma^2 - \psi]^2 + 9\gamma^2\}^{1/10} > 0, \quad \text{(A18)}$$

where e_E is given by Eq. (A14). The inequalities cited allow no ambiguities in the solution.

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